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RESEARCH ARTICLE

Šidák-type tests for the two-sample problem based on precedence and exceedance statistics

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This paper deals with a class of nonparametric two-sample tests for ordered alternatives. The test statistics proposed are based on the number of observations from one sample that precede or exceed a threshold specified by the other sample, and they are extensions of Šidák's test. We derive their exact null distributions and also discuss a large-sample approximation. We then study their power properties exactly against the Lehmann alternative and make some comparative comments. Finally, we present an example to illustrate the proposed tests.

Keywords: two-sample problem; exceedance statistics; precedence statistics; Lehmann alternative; stochastic ordering

AMS Subject Classification: 62G10; 62E15

1. Introduction

Suppose X and Y are random variables with absolutely continuous univariate distributions F and G , respectively. For testing the hypothesis $H_0 : F(x) = G(x)$ against the alternative

$$H_A : F(x) > G(x), \quad (1)$$

there are simple tests based on available precedences and exceedances. One can count the number of observations in the Y -sample above all observations in the X -sample, or the number of observations in the X -sample below all those in the Y -sample.

As suggested by Tukey [1], one or both of these statistics might be used to test H_0 against H_A in (1). The test based on the sum of these two quantities is mentioned as the earliest work of Šidák on nonparametric statistics; see [2]. The null distribution of this test statistic was studied by Šidák and Vondráček [3] and tables of critical values were produced by these authors. A slight modification of the test statistic based on the sum became popular as Tukey's Quick Test (see [4] and [5]). It basically leads to the same critical regions as Šidák's test. Hájek and Šidák [6] found that the same statistic

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also leads to locally most powerful rank tests for testing H_0 against a one-sided shift in the location parameter if the underlying distribution is uniform. They discussed some other test statistics based on exceeding observations, such as Haga's test [7] and E -test also discussed in [8]. In all these tests, the counts were with respect to the extreme order statistics from one or both samples.

The extreme sample values may get inflated by possible outliers, which may adversely affect the performance of these test statistics. For this reason, we may want to reduce their influence by defining thresholds above the smallest and below the largest observed values in the samples. Let X_1, \dots, X_m and Y_1, \dots, Y_n be independent random samples from continuous distributions F and G , respectively. Thresholds based on the $(r+1)$ -th order statistic from the Y -sample and $(m-s)$ -th order statistic from the X -sample define the exceedance and precedence statistics of the form

$$\begin{aligned} A_s &= \text{the number of } Y\text{-observations larger than } X_{(m-s)}, \\ B_r &= \text{the number of } X\text{-observations smaller than } Y_{(1+r)}, \end{aligned} \quad (2)$$

where $0 \leq s < m$ and $0 \leq r < n$.

In this paper, we propose a family of rank statistics for the two-sample problem in which the test statistic is a sum of A_s and B_r for appropriate choices of s and r . It includes Šidák's test as a special case.

Tests based on the number of precedences (B_r) were recommended by [9] for life-testing since a location shift can be effectively detected before all the data have been collected. They can be successfully applied to the general two-sample problem stated above. Some basic references on precedence tests include [10], [11], [12], [13], [14]. There are many extensions of precedence tests; see [15], [16], [17], [18], [19]. For more details on these developments, one may refer to Ng and Balakrishnan [20]. Recently, a family of tests based on the minimum of A_s and B_r has been studied in [21].

The rest of this paper is organized as follows. In Section 2, we introduce the new test statistics. In Section 3, we derive the exact null distributions of these test statistics and suggest some approximations for large samples. In Section 4, we derive the exact distributions of the test statistics under the Lehmann alternative and study the power functions of the tests against this alternative. In Section 5, we compare the powers of the proposed tests with other known tests based on exceedances, and also present an illustrative example. Proofs of the theorems are relegated to the Appendix.

2. The proposed test statistics

To test H_0 versus H_A in (1), we propose the test statistic

$$V_\rho = A_s + B_r, \quad (3)$$

where the threshold statistics $X_{(m-s)}$ and $Y_{(1+r)}$ are determined as $s = [\rho m]$ and $r = [\rho n]$ for some $0 \leq \rho < 1$, with $[\cdot]$ denoting the integer part. Various values of ρ yield a family of test statistics which we refer to as Šidák-type tests. Reasonable values of ρ are between 0 and 1/2. For $\rho = 0$, it is equivalent to Šidák's statistic [3]. $\rho > 0$ determines a part from the ordered samples that are skipped before the threshold is specified. Its role will be discussed in more detail later in Section 5.2.

Table 1. Lifetimes of two samples of an insulating fluid.

Group	Lifetimes									
X	0.49	0.64	0.82	0.93	1.08	1.99	2.06	2.15	2.57	4.75
Y	1.34	1.49	1.56	2.10	2.12	3.83	3.97	5.13	7.21	8.71

Table 2. Computation of V_ρ -statistic.

r	Y -threshold	Precedences	r	X -threshold	Exceedances
0	$Y_{(1)} = 1.34,$	$B_0 = 5,$	0	$X_{(10)} = 4.75,$	$A_0 = 3,$
1	$Y_{(2)} = 1.49,$	$B_1 = 5,$	1	$X_{(9)} = 2.57,$	$A_1 = 5,$
2	$Y_{(3)} = 1.56,$	$B_2 = 5,$	2	$X_{(8)} = 2.15,$	$A_2 = 5.$

Evidently, large values of V_ρ lead to the rejection of H_0 in favor of the stochastically ordered alternative in H_A . It is reasonable to select ρ to be small since we want to reduce the possible influence of a small number of potential outliers.

For equal sample sizes, the parameters s and r , specifying the threshold positions, are equal and in this case the contiguous order statistics determine the family of test statistics. For simplicity, let us denote the family of test statistics in this case by $V_r = A_r + B_r$ with $r = 0, 1, 2, \dots$

The following example is useful for an illustration of the proposed V_ρ -test statistic. The data is a subset of a data on breakdown times (in minutes) of an insulating fluid that is subjected to high voltage stress presented in [22]. Take X - and Y -samples to be Samples 3 and 6 from [22, p. 462], respectively.

Example 1. Ten units each of group X and group Y were placed simultaneously on a life-testing experiment, and their lifetimes (in minutes) were observed and are as presented in Table 1.

In this case, we have $m = n = 10$. Let $r = 0, 1$ and 2 and take consecutively the threshold values to be the pairs $(Y_{(1)}, X_{(10)})$, $(Y_{(2)}, X_{(9)})$, and $(Y_{(3)}, X_{(8)})$. We find the corresponding precedence and exceedance statistics as presented in Table 2. With these, the first three Šidák-type test statistics are readily found to be $V_0 = 8$, $V_1 = 10$ and $V_2 = 10$.

The tests from the family (3) have advantage to some of the other rank tests in the case when a small number of outliers are expected to be present in the data.

3. Null distribution

In this section, we derive the exact null distribution of the Šidák-type test statistic defined in (3), provide some tables of critical values for some selected small sample sizes, and finally suggest some approximation for large sample sizes.

3.1. Exact distribution

Given the joint distribution of A_s and B_r under the null hypothesis H_0 , the cumulative distribution function of V_ρ -statistic, for $0 \leq z \leq m + n$, is given by

$$P(V_\rho \leq z | F = G) = \sum_{i=0}^z \sum_{k=0}^{z-i} P(A_s = k, B_r = i | F = G). \quad (4)$$

THEOREM 3.1 For any $0 \leq s < m$ and $0 \leq r < n$, the joint probability mass function of

Table 3. Critical values of Šidák-type tests for $m = n = 6(1)25$ and different choices of r at 5% level of significance

r/n	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
0	5	5	5	5	5	5	5	5	5	5	5	6	6	6	6	6	6	6	6	6
1	8	8	8	8	8	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9
2	10	11	12	11	11	11	11	12	12	12	12	12	12	12	12	12	12	12	12	12
3	*	*	13	14	15	14	14	14	14	14	14	14	14	14	15	14	15	15	15	15
4	*	*	*	*	16	16	17	18	17	17	17	17	17	17	17	17	17	17	17	17
5	*	*	*	*	*	18	18	19	20	20	20	20	20	20	20	20	20	20	20	20
6	*	*	*	*	*	*	*	20	21	22	22	23	22	22	22	22	22	22	23	23
7	*	*	*	*	*	*	*	*	*	23	24	24	25	25	26	25	25	25	25	25
8	*	*	*	*	*	*	*	*	*	*	*	25	26	26	27	27	28	28	27	28
9	*	*	*	*	*	*	*	*	*	*	*	*	*	*	28	29	29	30	30	31
10	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	31	31	32	32

A_s and B_r , under $H_0 : F(x) = G(x)$, is given by

$$\begin{aligned}
 P(A_s = k, B_r = i) &= \frac{\binom{s+k}{s} \binom{r+i}{r}}{\binom{m+n}{n}} \binom{m+n-s-r-i-k-2}{n-r-k-1}, \\
 &\quad \text{for } 0 \leq i \leq m-s-1, \text{ and } 0 \leq k \leq n-r-1, \\
 &= \frac{\binom{m+n-r-i-1}{n-r-1} \binom{m+n-s-k-1}{m-s-1}}{\binom{m+n}{n}} \binom{k+i-m-n+s+r}{k-n+r}, \\
 &\quad \text{for } m-s \leq i \leq m, \text{ and } n-r \leq k \leq n, \\
 &= 0, \text{ otherwise.}
 \end{aligned}$$

The proof of this theorem is presented in the Appendix.

To compute the cumulative distribution function of V_ρ -statistic under H_0 , we just substitute for $P(A_s = k, B_r = i | F = G)$ from Theorem 3.1 into (4).

The cumulative distribution function in (4) is thus distribution-free. However, it does not take a simpler expression. For the simplest case when $\rho = 0$, Šidák and Vondráček [3] presented the formula

$$P(V_0 \leq z) = \left\{ \binom{m+n-z}{n} + \sum_{j=0}^{z-1} \binom{m+n-z-1}{m-j} \right\} / \binom{m+n}{n}.$$

3.2. Critical values

Using the exact null distribution in (4), we can determine the critical region of the test statistic V_ρ for a pre-fixed level of significance α . Under the alternative hypothesis that Y is stochastically larger than X as in (1), we expect the X -observations to take on most of the smaller ranks. Hence, H_0 is rejected if $V_\rho \geq c$, where critical value c is determined as the minimal c such that $P(V_\rho \geq c | H_0) \leq \alpha$.

For small sample sizes, the expression in (4) is easy to compute¹. Table 3 presents the critical values c of the V_r -tests for the choices of the sample sizes $m = n = 6, \dots, 25$ for $\alpha = 0.05$, where the index r corresponds to the threshold statistics $Y_{(1+r)}$ and $X_{(m-r)}$.

Note that due to the discreteness of the distributions of non-randomized test statistics based on ranks, the significance levels of the different V_r -tests are not the same. In order

¹These and further calculations have been carried out on a PC computer by using the statistical package R. The code can be provided by the corresponding author upon request.

Table 4. Critical values for $m = 40$ and $n = 20(4)40$ and different choices of s and r at 5% level of significance

ρ	m	s	n	r	c.v.	α_1	α_2	ρ	m	s	n	r	c.v.	α_1	α_2
0	40	0	20	0	7	0.043	0.068	0.15	40	6	20	3	20	0.044	0.060
0	40	0	24	0	7	0.030	0.050	0.15	40	6	24	3	19	0.042	0.058
0	40	0	28	0	6	0.034	0.069	0.15	40	6	28	4	20	0.048	0.067
0	40	0	32	0	6	0.034	0.061	0.15	40	6	32	4	20	0.044	0.061
0	40	0	36	0	6	0.032	0.058	0.15	40	6	36	5	22	0.037	0.051
0	40	0	40	0	6	0.032	0.058	0.15	40	6	40	6	23	0.041	0.056
0.05	40	2	20	1	12	0.041	0.059	0.2	40	8	20	4	24	0.044	0.057
0.05	40	2	24	1	11	0.043	0.064	0.2	40	8	24	4	23	0.041	0.055
0.05	40	2	28	1	11	0.034	0.053	0.2	40	8	28	5	24	0.043	0.058
0.05	40	2	32	1	10	0.048	0.075	0.2	40	8	32	6	25	0.047	0.064
0.05	40	2	36	1	10	0.048	0.075	0.2	40	8	36	7	27	0.039	0.052
0.05	40	2	40	2	12	0.041	0.062	0.2	40	8	40	8	28	0.043	0.056
0.1	40	4	20	2	16	0.044	0.063	0.25	40	10	20	5	28	0.042	0.053
0.1	40	4	24	2	15	0.044	0.062	0.25	40	10	24	6	28	0.049	0.065
0.1	40	4	28	2	15	0.037	0.053	0.25	40	10	28	7	29	0.048	0.062
0.1	40	4	32	3	16	0.046	0.066	0.25	40	10	32	8	30	0.049	0.064
0.1	40	4	36	3	16	0.044	0.064	0.25	40	10	36	9	32	0.040	0.051
0.1	40	4	40	4	18	0.036	0.052	0.25	40	10	40	10	33	0.043	0.055

to achieve the same level of significance for all tests under study, we use the randomized test procedure described below. This allows us to make meaningful and more reasonable comparison of their power performance.

In order to achieve the prescribed α for all tests, for each realization (labeled i -th, say) of two random samples, we calculate the probability P_i of rejecting H_0 as follows:

$$P_i = \begin{cases} 1, & \text{if } V_\rho \geq c \\ \frac{\alpha - \alpha_1}{\alpha_2 - \alpha_1}, & \text{if } V_\rho = c - 1 \\ 0, & \text{otherwise,} \end{cases} \quad (5)$$

where c is a possible critical value of the statistic V_ρ such that $P(V_\rho \geq c) = \alpha_1$, $P(V_\rho \geq c - 1) = \alpha_2$, with $\alpha_1 < \alpha < \alpha_2$. For example, Table 4 presents the exact levels of significance α_1 and α_2 (when $\alpha = 5\%$) of the V_ρ -tests for $m = 40$ and $n = 20(4)40$.

For the V_0 -test, Šidák and Vondráček [3] presented tables up to $m = 26$ and $n = 26$ at the 5% and 1% levels of significance.

For small values of m and n , the critical values and the exact significance probabilities of the V_ρ -test can be computed without any difficulty, as done in Table 3. However, for large sample sizes, this would require a heavy computational effort and time. For this reason, we present below some large-sample approximations for the null distributions of V_ρ -statistics.

3.3. Large-sample approximation

For the V_0 -test, Šidák and Vondráček [3] presented tables of approximate critical values for 5% and 1% levels of significance. As $m/n \rightarrow 1$, the right tail probability of the test statistic is asymptotically equivalent to $\frac{c+2}{2^{c+1}}$, where c is the corresponding critical value. For $\rho > 0$, the tail approximation by negative binomial distribution turns out to be reasonable. Since the majority of the probability mass of V_ρ is in the lower tail of the distribution, we calculate the upper tail as $1 - \sum_{i+k \leq z} P(A_s = k, B_r = i)$.

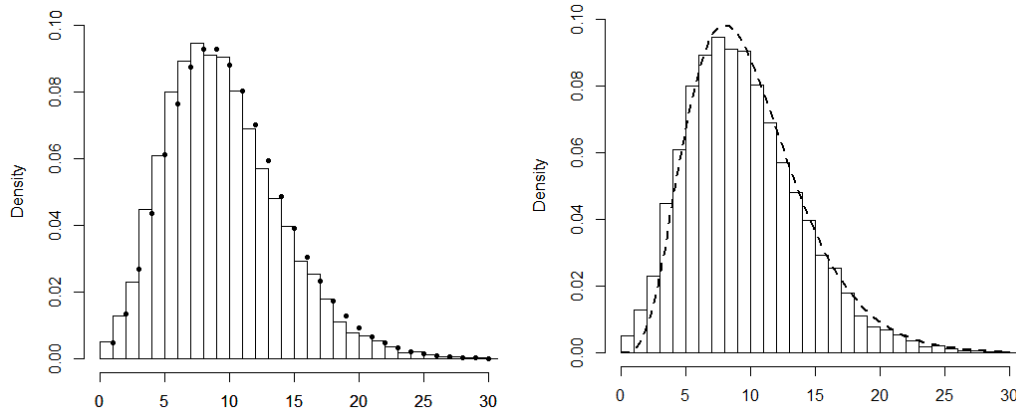


Figure 1. Negative binomial and chi-square approximations of V_ρ distribution for $m = 400$, $\rho = 0.01$.

Table 5. Values of $P(\chi^2_\nu > c)$ (near 5% critical values)

m	ρ	c.v.	χ^2 -approx.	m	ρ	c.v.	χ^2 -approx.
40	0	6	0.0497	100	0	6	0.0489
	0.05	12	0.0571		0.05	20	0.0649
	0.1	18	0.0496		0.1	33	0.0587
	0.15	23	0.0535		0.15	45	0.0585
	0.2	28	0.0538		0.2	58	0.0475
	0.25	33	0.0529		0.25	69	0.0513

THEOREM 3.2 As $m, n \rightarrow \infty$ and $m/n \rightarrow 1$,

$$P(V_\rho \leq z) = \sum_{i+k \leq z} P(V_\rho = z) \sim \sum_{i=0}^z \binom{2(s+1)+i-1}{i} 2^{-2(s+1)} 2^{-i},$$

where $s = [\rho m]$.

The approximating probability is then the value given by the c.d.f. of a negative binomial random variable with parameters $2(s+1)$ and $1/2$ (Figure 1, left). The proof of this theorem is presented in the Appendix.

The chi-square approximation (Figure 1, right) is also quite reasonable in the practical range of sample sizes (between 25 to 100) as long as n does not differ too much from m . In Table 5, we provide an example of the exact significance probabilities for the V_ρ -statistics (close to 5% level) for the choices of the sample size $m = n = 40$ and 100. It is given by a chi-square distribution with degrees of freedom $[\rho m] + 1$.

4. Distributions under alternatives

4.1. Distribution under Lehmann alternative

In this section, we derive an expression for the distribution of V_ρ under the Lehmann alternative given by

$$H_{LE} : G(x) = 1 - (1 - F(x))^{1/\eta}, \quad (6)$$

for some $\eta > 1$. When $\eta = 1$, the resulting distributions satisfy the null hypothesis H_0 , while $\eta > 1$ yields various distributions in the alternative hypothesis H_{LE} , with larger

values of η indicating stronger attraction towards $H_A : F(x) \geq G(x)$; see [23] for further discussion on this class of alternatives.

As in the derivation of the null distribution, the joint probability mass function of A_s and B_r under H_{LE} can be used for obtaining the distribution of V_ρ under H_{LE} .

Under the hypothesis H_{LE} in (6), the exact cumulative distribution function of the V_ρ -statistic, for $0 \leq z \leq m + n$, is given by (4) with the joint distribution of A_s and B_r now being under H_{LE} , as established in the following theorem.

THEOREM 4.1 *For any $0 \leq s < m$ and $0 \leq r < n$, the joint probability mass function of A_s and B_r , under H_{LE} in (6), is given by*

$$\begin{aligned} P(A_s = k, B_r = i) &= \frac{m!n!(1/\eta)}{r!s!(n-k-r-1)!k!} S_p S_z, \\ &\quad \text{for } 0 \leq i \leq m-s-1, \text{ and } 0 \leq k \leq n-r-1, \\ &= \frac{m!n!\eta}{(n-r-1)!(m-s-1)!(i-m+s)!(m-i)!} S'_p S'_z, \\ &\quad \text{for } m-s \leq i \leq m, \text{ and } n-r \leq k \leq n, \\ &= 0, \text{ otherwise,} \end{aligned}$$

where S_p , S_z , S'_p and S'_z are as follows:

$$\begin{aligned} S_p &= \sum_{p=0}^r (-1)^p \binom{r}{p} \frac{\Gamma(m-i+(n-r+p)/\eta)}{\Gamma(m+(n-r+p)/\eta+1)}, \\ S_z &= \sum_{z=0}^{n-k-r-1} (-1)^z \binom{n-k-r-1}{z} \frac{\Gamma(s+(z+k)/\eta+1)}{\Gamma(m-i+(z+k)/\eta+1)}, \\ S'_p &= \sum_{p=0}^{i-m+s} (-1)^p \binom{i-m+s}{p} \frac{\Gamma(n-r+(m-i+p)\eta)}{\Gamma(k+(m-i+p)\eta+1)}, \\ S'_z &= \sum_{z=0}^{m-s-1} (-1)^z \binom{m-s-1}{z} \frac{\Gamma(k+(z+s+1)\eta)}{\Gamma(n+(z+s+1)\eta+1)}. \end{aligned}$$

The proof of this theorem is presented in the Appendix.

Consequently, the distribution of V_ρ -statistic under H_{LE} is distribution-free as well.

4.2. Power against Lehmann alternative

Now, we demonstrate the use of the exact cumulative distribution function of V_ρ under Lehmann alternative as well as the Monte Carlo simulation method for the computation of the power of the V_ρ -test against this alternative. For this purpose, we generated 100,000 sets of data from F and $1 - (1 - F(x))^{1/\eta}$, respectively, and computed the test statistic V_ρ for each set. The power values were estimated by the rejection rates of the null hypothesis for different values of η .

To make meaningful comparison of the power values of different tests, we calculated power functions at prescribed exact level of significance α as follows. First, for any V_ρ -test,

Table 6. Power comparison of V_r -tests for $m = n = 10$ at 5% level of significance

V_r -test	$\eta = 2$	$\eta = 3$	$\eta = 4$	$\eta = 5$	$\eta = 6$	$\eta = 7$
V_0	0.3212	0.5799	0.7432	0.8415	0.8969	0.9318
V_1	0.3291	0.5854	0.7430	0.8370	0.8911	0.9219
V_2	0.3070	0.5536	0.7133	0.8114	0.8728	0.9097
V_3	0.2946	0.5384	0.7012	0.8020	0.8673	0.9064
V_4	0.3211	0.5801	0.7492	0.8468	0.9021	0.9375

Table 7. Power comparison of V_r -tests for $m = n = 20$ at 5% level of significance

V_r -test	$\eta = 2$	$\eta = 3$	$\eta = 4$	$\eta = 5$	$\eta = 6$	$\eta = 7$
V_0	0.4566	0.7859	0.9207	0.9705	0.9894	0.9952
V_1	0.5061	0.8292	0.9436	0.9808	0.9931	0.9974
V_2	0.5230	0.8379	0.9476	0.9818	0.9928	0.9969
V_3	0.5182	0.8355	0.9445	0.9795	0.9918	0.9957
V_4	0.5149	0.8262	0.9416	0.9774	0.9901	0.9956
V_5	0.4971	0.8137	0.9323	0.9742	0.9890	0.9948
V_6	0.4737	0.7934	0.9208	0.9692	0.9866	0.9936
V_7	0.4499	0.7684	0.9063	0.9618	0.9826	0.9919
V_8	0.4791	0.8061	0.9299	0.9737	0.9888	0.9955

we determine two values α_1 and α_2 such that

$$P(V_\rho \geq c) = \alpha_1 \quad \text{and} \quad P(V_\rho \geq c - 1) = \alpha_2,$$

where c is given by $P(V_\rho \geq c|H_0) \leq \alpha$, so that the interval (α_1, α_2) contains the critical level, say $\alpha = 0.05$. Next, we calculate the power values corresponding to the two critical values c and $c - 1$ as

$$\beta_1 = P(V_\rho \geq c|H_{LE}) \quad \text{and} \quad \beta_2 = P(V_\rho \geq c - 1|H_{LE}).$$

Then, the power of the test at exact level α is estimated by

$$\beta = \pi\beta_2 + (1 - \pi)\beta_1,$$

where $\pi = \frac{\alpha - \alpha_1}{\alpha_2 - \alpha_1}$ is the adjusting factor used in the randomization procedure in (5).

For $m = n = 10$ and $\eta = 2(1)7$, the power values of the V_ρ -tests corresponding to $r = 0, \dots, 4$, against the Lehmann alternative H_{LE} in (6), are presented in Table 6, where the significance level is set as $\alpha = 0.05$. Similarly, for $m = n = 20$, the power values of the V_ρ -tests corresponding to $r = 0, \dots, 8$, are presented in Table 7.

From Tables 6 and 7, we see that the power values of all tests increase with increasing η . The power of V_0 (original Šidák test) is much less than the power of the next two V_r -tests for sample size $m = 10$, and much less than the power of the next four V_r -tests for sample size $m = 20$. For each of the six fixed values 2 (1) 7 of η , the power increases up to the third V_r -test, showing that the V_0 -test, based on the extremal thresholds, is less powerful than the tests based on the next extremal thresholds pairs $(Y_{(2)}, X_{(m-1)})$ and $(Y_{(3)}, X_{(m-2)})$.

For unequal sample sizes, we compare the power functions for fixed $\eta = 2$; for other values of $\eta > 1$, we observed a similar behavior and so we do not present the corresponding results for conciseness. Table 8 provides the power values for $m = 40$ and $n = 20, 24, 28, 32, 36, 40$. The proportion coefficient ρ specifies the six V_ρ -tests. The power functions were estimated through Monte Carlo simulations, with 100,000 simulated data sets for each case.

Table 8. Power of V_ρ -test against $H_1 : G = 1 - (1 - F)^{1/2}$ for $m = 40$ and $n = 20(4)40$ at 5% level of significance

proportion (ρ)	Second sample size (n)					
	20	24	28	32	36	40
0	0.3472	0.4147	0.4771	0.5275	0.5685	0.6016
0.05	0.4647	0.5242	0.5995	0.6548	0.7010	0.7367
0.1	0.5161	0.5784	0.6313	0.6910	0.7276	0.7708
0.15	0.5489	0.5960	0.6618	0.6937	0.7425	0.7750
0.2	0.5579	0.5990	0.6586	0.7066	0.7428	0.7703
0.25	0.5675	0.6203	0.6669	0.7012	0.7334	0.7625

Table 9. Power of V_ρ -test against $H_1 : G = 1 - (1 - F)^{1/2}$ for $m = 100$ and $n = 50(10)100$ at 5% level of significance

proportion (ρ)	Second sample size (n)					
	50	60	70	80	90	100
0	0.5002	0.5805	0.6658	0.7137	0.7611	0.7858
0.05	0.7492	0.8404	0.8936	0.9261	0.9502	0.9646
0.1	0.8205	0.8806	0.9305	0.9569	0.9716	0.9813
0.15	0.8296	0.9030	0.9406	0.9618	0.9750	0.9852
0.2	0.8555	0.9176	0.9481	0.9595	0.9781	0.9825
0.25	0.8615	0.9189	0.9439	0.9652	0.9750	0.9809

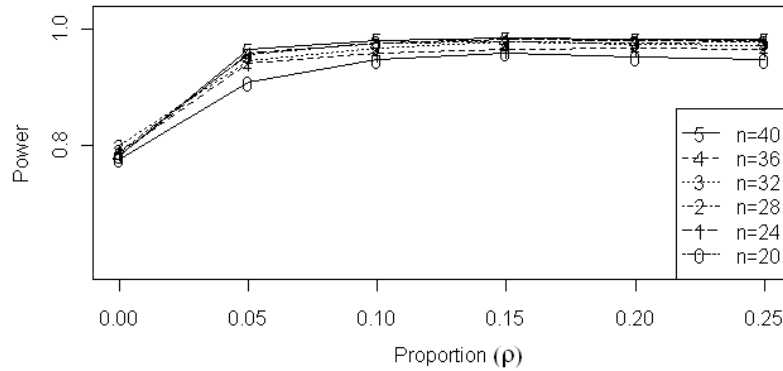


Figure 2. Power functions of V_ρ -tests for $m = 40$ and $n = 20, 24, 28, 32, 36, 40$ against the Lehmann alternative with $\eta = 2$ at 5% level of significance.

Figure 2 illustrates the gain in power of using any of the first five V_ρ -tests with $\rho > 0$ instead of Šidák's V_0 -test. Table 9 provides similar results for $m = 100$ and six values for n corresponding to six values of the proportion coefficient ρ .

5. Discussion

5.1. Remark on consistency of the test

Since the distribution of the V_ρ -test under H_{LE} is distribution-free, any particular underlying distribution F can be used to prove the test consistency. Sen [24] has proved (in above notation) the following: For $G(x) = F(x - \theta)$ with $H_0 : \theta = 0$, the test based on A_s is consistent against the set of alternatives $H_A : \theta > 0$ for F belonging to the domain of attraction for maxima of the Gumbel (type 1) family of cdf's. Consequently, the test based on the sum of A_s and B_r is consistent for this family of cdf's.

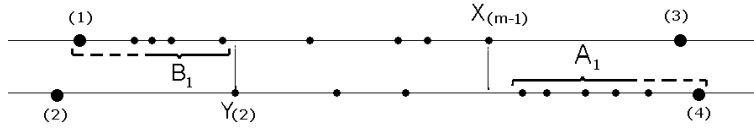


Figure 3. Outliers present in data.

For our purpose, let F be a Gumbel distribution, i.e.,

$$F(x) = 1 - \exp(-e^x)$$

for $-\infty < x < \infty$. This distribution belongs to the above mentioned family of cdf's.

The location shift alternative G given by $G(x) = F(x - \theta)$ in this case is a Lehmann alternative of the form (6), with $1/\eta = e^{-\theta}$. Using Sen's result, we may conclude that the V_ρ -test is consistent against Lehmann alternatives.

5.2. Outlier-inflated distribution

Suppose there are a small number (say, less than 10%) of “spurious” values in the observed data set. Let us consider the following example.

Example 2. In Figure 3, points labeled by (1) and (3) in the X -sample and (2) and (4) in the Y -sample lie away from the majority of observed data, i.e., they are potential outliers; see [25] for a thorough discussion on outliers.

For the hypothesis testing problem $H_0 : F(x) = G(x)$ against $H_A : F(x) > G(x)$, we might want to apply some quick tests like $V_0 = A_0 + B_0$ or $V_1 = A_1 + B_1$. Outliers like (1) and (4) do not add much to the test statistic V_0 ; here, $B_0 = 0$ and $A_0 = 1$, while outliers like (2) and (3) inflate the thresholds and might significantly decrease B_0 and/or A_0 . For this reason, the V_ρ -test with $\rho > 0$ may be better since it is robust to the presence of a small number of outliers in the data.

Example 3. Let us now consider the data arising from a contaminated distribution of the form

$$F_\varepsilon = (1 - \varepsilon)F + \varepsilon F_c,$$

where ε specifies a small part of contamination with distribution F_c . Let the distribution of the second sample similarly be

$$G_\varepsilon = (1 - \varepsilon)G + \varepsilon G_c.$$

To allow 5% outliers in this setup, we generated samples from contaminated normal distributions as follows:

$$\begin{aligned} X &\sim F_\varepsilon = 0.95 N(5, 1) + 0.05 N(8, 1), \\ Y &\sim G_\varepsilon = 0.95 N(6, 1) + 0.05 N(3, 1). \end{aligned}$$

The two distributions are plotted in Figure 4.

For testing $H_0 : F(x) = G(x)$ vs $H_1 : F(x) > G(x)$, we use the V_ρ -tests with $\rho = 0, 0.05, 0.1, 0.15, 0.2, 0.25$. The corresponding threshold values s and r are $s = \lfloor \rho m \rfloor$ and $r = \lfloor \rho n \rfloor$, respectively. Simulating 100 observations from each distribution, we calculated the test statistics (see Table 10).

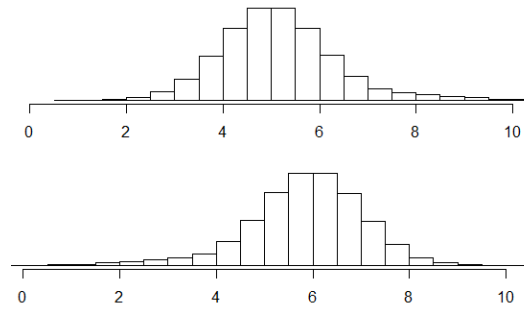


Figure 4. Contaminated data.

Table 10. Test results for contaminated normal data and comparison of V_ρ -tests. (near 5% critical values)

$r = [\rho m]$	V_ρ	crit. value	
0	1	6	Do not reject
5	19	20	
10	33	33	
15	66	45	Reject
20	82	58	
25	96	69	

As we can expect, the first V_ρ -tests are not robust to the presence of outliers. More specifically, the V_ρ -test for $\rho = 0, 0.05, 0.1$ would not reject H_0 at 5% level of significance, while the V_ρ -test for $\rho = 0.15, 0.2, 0.25$ suppresses the effect of outliers and do indeed reject H_0 .

Depending on the expected percentage of contaminated data, we could recommend to use a V_ρ -test with a suitable choice of ρ . Clearly, such a test will reject the null hypothesis more precisely when it is not true and its power will be similar to the power of other tests from the family. Therefore, in the case when some percentage of outliers is expected, the use of Šidák-type tests would be recommended.

5.3. Comparative comments

In this section, we discuss briefly several nonparametric exceedance-type tests from the literature, and compare the proposed Šidák-type tests with these tests through an example. For more details about these tests, we refer the readers to [20].

The classical precedence test and the maximal precedence test are useful in the case of life-testing experiments wherein data become available naturally in order of size. However, they can be used for testing $H_0 : F(x) = G(x)$ against the stochastically ordered alternative as well.

- For fixed $0 \leq r \leq n$, the classical precedence test P_r is simply (in terms of exceedance statistics defined by (2)) the number of failures from the X -sample before the $(r+1)$ -th failure from the Y -sample;
- The maximal precedence statistic Q_r has been defined by [26] as the maximum number of failures occurring from the X -sample before the first, between the first and the second, \dots , and between the r -th and $(r+1)$ -th failures from the Y -sample;
- The M_r -test statistic is given by

$$M_r = \max\{n - A_s, m - B_r\},$$

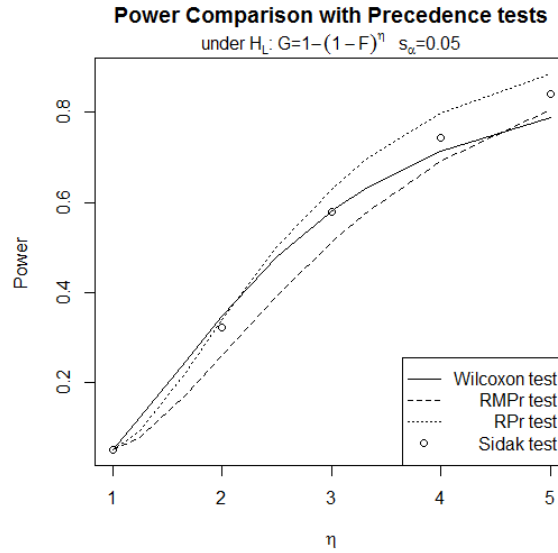


Figure 5. Power comparison of V_4 -test with precedence tests against Lehmann alternatives with $\eta = 2$ to 5 at 5% level of significance.

Table 11. Different test statistics and p -values for the insulating fluid data

r	P_r	p-value	Q_r	p-value	M_r	p-value	V_r	p-value
0	5	0.0163	5	0.0163	7	0.0186	8	0.01054
1	5	0.0704	5	0.0325	5	0.0177	10	0.02826
2	5	0.1749	5	0.0487	5	0.0795	10	0.10847

and it was recently introduced by [21]. It generalizes in some sense the E -test of [6];

- The Wilcoxon's rank-sum statistic WR is known to provide a good nonparametric test for the hypothesis testing problem described above against the alternative $H_1 : F(x) > G(x)$. Its test statistic is based on the sum of the ranks of observations from one of the samples obtained from the combined sample.

The power of a V_4 -test is compared with the power of $RPr(r)$ and $RMPPr(r)$ for the case $r = 4$ and $m = n = 10$. The computations here were carried for $RPr(r)$ and $RMPPr(r)$ tests through $\eta = 2$ to 5 at $\alpha = 0.05$. The plots are given in Figure 5. Clearly, the power of the V_r -test is similar to the power of the two precedence-type tests and the Wilcoxon rank-sum test. Therefore, in the case of Lehmann alternatives, the use of Šidák-type tests would be recommended.

In the following example, we compare the V_r -tests with the precedence test P_r , the maximal precedence test Q_r , and the M_r -test described above.

Example 4. Considering the same data as in Example 1 (see Table 1), we can carry out a nonparametric test for the hypothesis $H_0 : F(x) = G(x)$ through the first four tests from each of the above families of tests. Table 11 provides the values of the test statistics and the corresponding p -values.

In this example, the first five smallest X -values occurred before the smallest Y -value, and in addition, the last three largest Y -values occurred after the largest X -value. All the tests with $r = 0$ perform similarly, giving evidence against H_0 at the usual 5% level of significance. If M_0 or V_0 test is used, the data would provide strong evidence to reject H_0 . However, if P_1 -test had been used instead, it would not reject H_0 while the Q_1 , M_1 and V_1 tests all would reject H_0 . For $r \geq 2$, all tests provide similar conclusions.

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Appendix A. Proofs of Theorems

Proofs of Theorems 3.1 and 4.1

Let X_1, \dots, X_m and Y_1, \dots, Y_n be two independent random samples from continuous distributions F and G , respectively. For $0 \leq s < n$ and $0 \leq r < m$, let A_s and B_r be the statistics as defined in (2).

Consider k exceedances in the Y -sample with respect to $X_{(m-s)}$ and i precedences in the X -sample with respect to $Y_{(1+r)}$. First, suppose that $1 \leq k \leq n - r - 2$ and $0 \leq i \leq n - r - 1$. Event $\{A_s = k\}$ means that the $(m - s)$ -th ordered observation from the X -sample is between the $(n - k)$ -th and $(n - k + 1)$ -th ordered observations from the Y -sample, while event $\{B_r = i\}$ means that the $(1 + r)$ -th ordered observation from the Y -sample is between the i -th and $(i + 1)$ -th ordered observations from the X -sample. The first two cases in each of Theorems 3.1 and 4.1 arise according to the ordering of $Y_{(1+r)}$ and $X_{(m-s)}$:

$$\begin{aligned} &\text{If } Y_{(1+r)} < X_{(m-s)}, \text{ then } B_r \leq m - s - 1 \text{ and } A_s \leq n - r - 1, \\ &\text{while if } Y_{(1+r)} > X_{(m-s)}, \text{ then } B_r \geq m - s \text{ and } A_s \geq n - r. \end{aligned} \quad (\text{A1})$$

Here, we derive $P(A_s = k, B_r = i)$ in the case $Y_{(1+r)} < X_{(m-s)}$ for arbitrary absolutely continuous distributions F and G , and for the null hypothesis $F = G$ and then for the Lehmann alternative in (6).

Conditional on the Y -observations

$$Y_{(1+r)} = y_1, Y_{(n-k)} = y_2, Y_{(n-k+1)} = y_3, \quad (\text{A2})$$

define the event $W_{q,t}$ on the X -sample as follows:

$$W_{q,t} := \begin{cases} i & X\text{-observations preceding } y_1 \\ t & X\text{-observations between } y_1 \text{ and } y_2 \\ m - i - q - t & X\text{-observations between } y_2 \text{ and } y_3 \\ q & X\text{-observations exceeding } y_3, \end{cases}$$

where $0 \leq q \leq s$ and $0 \leq t \leq m - i - s - 1$.

The probability of $W_{q,t}$ is evidently given by the multinomial probability

$$\frac{m!}{i!t!(m-i-q-t)!q!} [F(y_1)]^i [F(y_2) - F(y_1)]^t [F(y_3) - F(y_2)]^{m-i-q-t} [1 - F(y_3)]^q, \quad (\text{A3})$$

for $y_1 < y_2 < y_3$. The conditional probability of $\{A_s = k, B_r = i\}$, given (A2), is obtained by summing (A3) over all $q = 0, \dots, s$ and $t = 0, \dots, m-i-s-1$. Hence, the unconditional probability of $\{A_s = k, B_r = i\}$, with respect to the joint distribution of $Y_{(1+r)}$, $Y_{(n-k)}$ and $Y_{(n-k+1)}$, is

$$\begin{aligned} P(A_s = k, B_r = i) &= \sum_{q=0}^s \sum_{t=0}^{m-i-s-1} \frac{m!}{i!t!(m-i-q-t)!q!} \int_{-\infty}^{\infty} \int_{y_1}^{\infty} \int_{y_2}^{\infty} [F(y_1)]^i [F(y_2) - F(y_1)]^t \\ &\quad \times [F(y_3) - F(y_2)]^{m-i-q-t} [1 - F(y_3)]^q g_*(y_1, y_2, y_3) dy_3 dy_2 dy_1, \end{aligned} \quad (\text{A4})$$

where g_* is the joint density function of the three order statistics $Y_{(1+r)}$, $Y_{(n-k)}$ and $Y_{(n-k+1)}$, from the Y -sample given by (see [27] or [28])

$$\begin{aligned} g_*(y_1, y_2, y_3) &= \frac{n!}{r!(n-k-r-2)!(k-1)!} [G(y_1)]^r [G(y_2) - G(y_1)]^{n-k-r-2} \\ &\quad \times [1 - G(y_3)]^{k-1} g(y_1)g(y_2)g(y_3), \quad \text{for } y_1 < y_2 < y_3, \end{aligned} \quad (\text{A5})$$

with g being the density corresponding to G .

Proof of Theorem 4.1

Under the Lehmann alternative in (6), the two distributions satisfy the relationships $(1 - G) = (1 - F)^{1/\eta}$ and $g(x) = (1/\eta)[1 - F(x)]^{(1/\eta)-1} f(x)$, with f and g being the densities corresponding to F and G , respectively. Substituting these in (A4) and then using (A5), we obtain

$$\begin{aligned} P(A_s = k, B_r = i | H_{LE}) &= \sum_{q=0}^s \sum_{t=0}^{m-i-s-1} C \int_{-\infty}^{\infty} \int_{y_1}^{\infty} \int_{y_2}^{\infty} [F(y_1)]^i [F(y_2) - F(y_1)]^t \\ &\quad \times [F(y_3) - F(y_2)]^{m-i-q-t} [1 - F(y_3)]^q [1 - (1 - F(y_1))^{1/\eta}]^r [(1 - F(y_1))^{1/\eta} \\ &\quad - (1 - F(y_2))^{1/\eta}]^{n-k-r-2} [1 - F(y_3)]^{(k-1)/\eta} [1 - F(y_1)]^{(1/\eta)-1} \\ &\quad \times [1 - F(y_2)]^{(1/\eta)-1} [1 - F(y_3)]^{(1/\eta)-1} f(y_1)f(y_2)f(y_3) dy_3 dy_2 dy_1, \end{aligned}$$

where $C = \frac{m!n!(1/\eta)^3}{i!t!(m-i-q-t)!q!r!(n-k-r-2)!(k-1)!}$.

Changing variables in the integral by $u_i = 1 - F(y_i)$, $i = 1, 2, 3$, together with $du_i =$

$-f(y_i)dy_i$, we get

$$\begin{aligned}
& P(A_s = k, B_r = i | H_{LE}) \\
&= \sum_{q=0}^s \sum_{t=0}^{m-i-s-1} C \int_0^1 \int_0^{u_1} \int_0^{u_2} (1-u_1)^i (u_1-u_2)^t (u_2-u_3)^{m-i-q-t} u_3^q \\
&\quad \times (1-u_1^{1/\eta})^r (u_1^{1/\eta} - u_2^{1/\eta})^{n-k-r-2} u_3^{(k-1)/\eta} u_1^{1/\eta-1} u_2^{1/\eta-1} u_3^{1/\eta-1} du_3 du_2 du_1 \\
&= \sum_{q=0}^s \sum_{t=0}^{m-i-s-1} C \sum_{z=0}^{n-k-r-2} (-1)^z \binom{n-k-r-2}{z} \sum_{p=0}^r (-1)^p \binom{r}{p} \int_0^1 \int_0^{u_1} \int_0^{u_2} (1-u_1)^i \\
&\quad \times (u_1-u_2)^t (u_2-u_3)^{m-i-q-t} u_3^q u_1^{p/\eta} u_2^{z/\eta} u_1^{(n-k-r-2-z)/\eta} u_3^{(k-1)/\eta} \\
&\quad \times u_1^{1/\eta-1} u_2^{1/\eta-1} u_3^{1/\eta-1} du_3 du_2 du_1, \tag{A6}
\end{aligned}$$

where in the last expression we have used binomial expansions for the power terms $(u_1^{1/\eta} - u_2^{1/\eta})^{n-k-r-2}$ and $(1-u_1^{1/\eta})^r$.

Then the integral in (A6) is simplified by the substitution $w = u_3/u_2$ with $du_3 = u_2 dw$ and further by $w = u_2/u_1$ with $du_2 = u_1 dw$, yielding

$$\begin{aligned}
J &= \int_0^1 \int_0^{u_1} \int_0^{u_2} (1-u_1)^i (u_1-u_2)^t (u_2-u_3)^{m-i-q-t} u_1^{(n-k-r-1-z+p)/\eta-1} u_2^{(z+1)/\eta-1} \\
&\quad \times u_3^{q+k/\eta-1} du_3 du_2 du_1 \\
&= \int_0^1 \int_0^{u_1} \int_0^1 (1-u_1)^i (u_1-u_2)^t u_2^{m-i-q-t} (1-w)^{m-i-q-t} u_1^{(n-k-r-1-z+p)/\eta-1} \\
&\quad \times u_2^{(z+1)/\eta-1} u_2^{q+k/\eta-1} w^{q+k/\eta-1} u_2 dw du_2 du_1 \\
&= B(q+k/\eta, m-i-q-t+1) \int_0^1 \int_0^{u_1} (1-u_1)^i (u_1-u_2)^t u_2^{m-i-t+(k+z+1)/\eta-1} \\
&\quad \times u_1^{(n-k-r-1-z+p)/\eta-1} du_2 du_1 \\
&= B(q+k/\eta, m-i-q-t+1) \int_0^1 \int_0^1 (1-u_1)^i u_1^t (1-w)^t u_1^{m-i-t+(k+z+1)/\eta-1} \\
&\quad \times w^{m-i-t+(k+z+1)/\eta-1} u_1^{(n-k-r-1-z+p)/\eta-1} u_1 dw du_1 \\
&= B(q+k/\eta, m-i-q-t+1) B(m-i-t+(k+z+1)/\eta, t+1) \\
&\quad \times B(m-i+(n-r+p)/\eta, i+1),
\end{aligned}$$

where $B(a, b) = \int_0^1 t^{a-1} (1-t)^{b-1} dt$ denotes the complete beta function.

Now substituting the above expression for J in (A6) and expressing the beta functions

through gamma functions, we obtain

$$\begin{aligned}
P(A_s = k, B_r = i | H_{LE}) &= \sum_{z=0}^{n-k-r-2} (-1)^z \binom{n-k-r-2}{z} \sum_{p=0}^r (-1)^p \binom{r}{p} \\
&\times \sum_{q=0}^s \sum_{t=0}^{m-i-s-1} \frac{m!n!(1/\eta)^3}{q!r!(n-k-r-2)!(k-1)!} \times \frac{\Gamma(q+k/\eta)}{\Gamma(m-i-t+k/\eta+1)} \\
&\times \frac{\Gamma(m-i-t+(k+z+1)/\eta)}{\Gamma(m-i+(k+z+1)/\eta+1)} \frac{\Gamma(m-i+(n-r+p)/\eta)}{\Gamma(m+(n-r+p)/\eta+1)} \\
&= \frac{m!n!(1/\eta)^3}{r!(n-k-r-2)!(k-1)!} \left(\sum_{p=0}^r (-1)^p \binom{r}{p} \frac{\Gamma(m-i+(n-r+p)/\eta)}{\Gamma(m+(n-r+p)/\eta+1)} \right) \\
&\times \left(\sum_{q=0}^s \frac{\Gamma(q+k/\eta)}{q!} \right) \sum_{z=0}^{n-k-r-2} (-1)^z \binom{n-k-r-2}{z} \frac{1}{\Gamma(m-i+(k+z+1)/\eta+1)} \\
&\times \left(\sum_{t=0}^{m-i-s-1} \frac{\Gamma(m-i-t+(k+z+1)/\eta)}{\Gamma(m-i-t+k/\eta+1)} \right).
\end{aligned}$$

The sums in the above expression were simplified as follows. The sum over t is

$$\begin{aligned}
Q_1 &= \sum_{t=0}^{m-i-s-1} \frac{\Gamma(m-i+(k+z+1)/\eta-t)}{\Gamma(m-i+k/\eta+1-t)} \\
&= -\frac{\eta}{(z+1)} \left[\frac{\Gamma(s+(k+z+1)/\eta+1)}{\Gamma(s+k/\eta+1)} - \frac{\Gamma(m-i+(k+z+1)/\eta+1)}{\Gamma(m-i+k/\eta+1)} \right],
\end{aligned}$$

where we used the identity $\sum_{k=0}^n \frac{\Gamma(a-k)}{\Gamma(b-k)} = \frac{1}{b-a-1} \left[\frac{\Gamma(a-n)}{\Gamma(b-n-1)} - \frac{\Gamma(a+1)}{\Gamma(b)} \right]$ for $a > n$ and $b > n+1$, $b \neq a+1$ [29, p. 492].

Next, we used the identity $\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{1}{k+p} = \frac{1}{p} \binom{n+p}{n}^{-1}$ [see 29, p. 498] so that the sum over z can be simplified as

$$\begin{aligned}
Q_2 &= \sum_{z=0}^{n-k-r-2} (-1)^z \binom{n-k-r-2}{z} \frac{1}{\Gamma(m-i+(k+z+1)/\eta+1)} Q_1 \\
&= \left[\sum_{z=0}^{n-k-r-2} (-1)^z \binom{n-k-r-2}{z} \frac{1}{(z+1)/\eta} \right] \frac{1}{\Gamma(m-i+k/\eta+1)} - \frac{1}{\Gamma(s+k/\eta+1)}
\end{aligned}$$

$$\begin{aligned}
& \times \left[\sum_{z=0}^{n-k-r-2} (-1)^z \binom{n-k-r-2}{z} \frac{\Gamma(s+(z+1+k)/\eta+1)}{\Gamma(m-i+(z+1+k)/\eta+1)} \frac{1}{(z+1)/\eta} \right] \\
& = \frac{\eta}{\Gamma(m-i+k/\eta+1)} \sum_{z=0}^{n-k-r-2} (-1)^z \binom{n-k-r-2}{z} \frac{1}{z+1} - \frac{\eta}{\Gamma(s+k/\eta+1)} \\
& \quad \times \frac{1}{(n-k-r-1)} \sum_{z=0}^{n-k-r-2} (-1)^z \binom{n-k-r-1}{z+1} \frac{\Gamma(s+(z+1+k)/\eta+1)}{\Gamma(m-i+(z+1+k)/\eta+1)} \\
& = \frac{\eta}{\Gamma(s+k/\eta+1)(n-k-r-1)} \sum_{z=0}^{n-k-r-1} (-1)^z \binom{n-k-r-1}{z} \\
& \quad \times \frac{\Gamma(s+(z+k)/\eta+1)}{\Gamma(m-i+(z+k)/\eta+1)},
\end{aligned}$$

and similarly the sum over q can be simplified as

$$Q_3 = \sum_{q=0}^s \frac{\Gamma(q+k/\eta)}{q!} = \frac{\eta}{k} \frac{\Gamma(s+k/\eta+1)}{s!},$$

where we have used the identity $\sum_{k=0}^n \frac{\Gamma(k+a)}{\Gamma(k+1)} = \frac{1}{a} \frac{\Gamma(n+a+1)}{\Gamma(n+1)}$ for any $a > 0$.

Thus, we obtain

$$\begin{aligned}
& P(A_s = k, B_r = i | H_{LE}) \\
& = \frac{m!n!(1/\eta)^3 \Gamma(s+k/\eta+1) \eta}{r!(n-k-r-2)!(k-1)!ks!} \frac{\eta}{\Gamma(s+k/\eta+1)} \frac{1}{(n-k-r-1)} S_p S_z \\
& = \frac{m!n!(1/\eta)}{r!s!(n-k-r-1)!k!} S_p S_z,
\end{aligned} \tag{A7}$$

where S_p and S_z are given by

$$\begin{aligned}
S_p & = \sum_{p=0}^r (-1)^p \binom{r}{p} \frac{\Gamma(m-i+(n-r+p)/\eta)}{\Gamma(m+(n-r+p)/\eta+1)}, \\
S_z & = \sum_{z=0}^{n-k-r-1} (-1)^z \binom{n-k-r-1}{z} \frac{\Gamma(s+(z+k)/\eta+1)}{\Gamma(m-i+(z+k)/\eta+1)}.
\end{aligned}$$

We have thus derived the first case of Theorem 4.1 for $0 \leq i \leq m-s-1$ and $2 \leq k \leq n-r-2$. It can be easily extended for $k=0$ and $k=n-r-1$ by using the joint density of two order statistics from distribution G .

In the second case of Theorem 4.1, when $m-s \leq i \leq n$ and $n-r \leq k \leq n$, the ordering of the observations can be viewed as a symmetric image of the ordering for the first case with the following switches: $F \leftrightarrow G$; $(r+1) \leftrightarrow (m-s)$; $i \leftrightarrow (n-k)$; $k \leftrightarrow (m-i)$.

So, if we denote $\psi(m, n, s, r, \eta, k, i)$ to be the RHS of (A7), i.e.,

$$\psi(m, n, s, r, \eta, k, i) = \frac{m!n!(1/\eta)}{r!s!(n-k-r-1)!k!} S_p S_z,$$

then for $m - s \leq i \leq n$ and $n - r \leq k \leq n$, we have

$$\begin{aligned} P(A_s = k, B_r = i | H_{LE}) &= \psi(n, m, n - r - 1, m - s - 1, 1/\eta, m - i, n - k) \\ &= \frac{m!n!\eta}{(n - r - 1)!(m - s - 1)!(i - m + s)!(m - i)!} S'_p S'_z, \end{aligned} \quad (A8)$$

where S'_p and S'_z are as stated in the theorem.

The last case of the theorem follows trivially due to (A1).

Proof of Theorem 3.1

The proof of the theorem follows readily by substituting $\eta = 1$ in Theorem 4.1. Hence, for $0 \leq i \leq m - s - 1$ and $0 \leq k \leq n - r - 1$, the sums S_p and S_z have closed-forms and after simplification, they become

$$\begin{aligned} S_{p0} &= \frac{1}{(i+1)!} \sum_{p=0}^r (-1)^p \binom{r}{p} \binom{m+n-r+p}{i+1}^{-1} \\ &= \frac{1}{(i+1)!} \frac{i+1}{r+i+1} \binom{m+n}{m+n-r-i-1}^{-1} = \frac{(m+n-r-i-1)!(r+i)!}{i!(m+n)!}, \\ S_{z0} &= \frac{1}{(m-i-s)!} \sum_{z=0}^{n-k-r-1} (-1)^z \binom{n-k-r-1}{z} \binom{m-i+k+z}{m-i-s}^{-1} \\ &= \frac{(k+s)!(m+n-s-r-i-k-2)!}{(m-s-i-1)!(m+n-i-r-1)!}, \end{aligned}$$

where we have used the identity [see 29, p. 509]

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{k+m}{l}^{-1} = \frac{l}{n+l} \binom{m+n}{m-l}^{-1}.$$

Substituting S_{p0} and S_{z0} in (A7) and by selecting $\eta = 1$, we obtain

$$\begin{aligned} P(A_s = k, B_r = i | H_0) &= \frac{m!n!}{r!(n-k-r-1)!s!k!} \\ &\times \frac{(m+n-r-i-1)!(r+i)!}{i!(m+n)!} \frac{(k+s)!(m+n-s-r-i-k-2)!}{(m-s-i-1)!(m+n-i-r-1)!} \\ &= \frac{\binom{s+k}{s} \binom{r+i}{r}}{\binom{m+n}{n}} \binom{m+n-s-r-i-k-2}{n-r-k-1}. \end{aligned}$$

Similarly, for $m - s \leq i \leq m$ and $n - r \leq k \leq n$, the sums S'_p and S'_z become

$$\begin{aligned} S'_{p0} &= \frac{(m-i+n-r-1)!(k+s-m+i-n+r)!}{(k-n+r)!(k+s)!}, \\ S'_{z0} &= \frac{(k+s)!(m+n-s-k-1)!}{(n-k)!(m+n)!}, \end{aligned}$$

and consequently, we get

$$\begin{aligned}
 P(A_s = k, B_r = i | H_0) &= \frac{m!n!}{(n-r-1)!(m-s-1)!(i-m+s)!(m-i)!} \\
 &\times \frac{(m-i+n-r-1)!(k+s-m+i-n+r)!(k+s)!(m+n-s-k-1)!}{(k-n+r)!(k+s)!(n-k)!(m+n)!} \\
 &= \frac{\binom{m+n-r-i-1}{n-r-1} \binom{m+n-s-k-1}{m-s-1}}{\binom{m+n}{n}} \binom{k+i-m-n+s+r}{k-n+r}.
 \end{aligned}$$

The last case of the theorem once again follows trivially due to (A1).

Proof of Theorem 3.2

For fixed sample sizes m and n , the lower tail of the exact distribution of V_ρ is represented by $\sum_{i+k \leq z} Q(k, i)$, where $Q(k, i) = P(A_s = k, B_r = i | F = G)$.

As $m, n \rightarrow \infty$ and $m/n \rightarrow 1$, the behavior of $\frac{\binom{m+n-s-r-i-k-2}{n-r-k-1}}{\binom{m+n}{n}}$ in $Q(k, i)$ is asymptotically equivalent to $2^{-(k+i+r+s+2)}$. Therefore, the large-sample approximation of $\sum_{i+k \leq z} Q(k, i)$ is given by

$$\begin{aligned}
 \sum_{i+k \leq z} Q(k, i) &\sim \sum_{i+k \leq z} \binom{s+k}{s} \binom{r+i}{r} 2^{-(k+i+r+s+2)} \\
 &= \sum_{i=0}^z \binom{s+i}{s} 2^{-(i+s+1)} \sum_{k=0}^{z-i} \binom{s+k}{s} 2^{-(s+1)} 2^{-k}.
 \end{aligned}$$

The last sum represents the distribution function of a negative binomial random variable ξ with parameters $s+1$ and $1/2$. Using the well-known relationship between negative binomial distribution and binomial distribution [30], we have $P(\xi \leq z) = P(\eta > s)$, where η has binomial distribution with parameters $z+s+1$ and $1/2$. Thus,

$$\begin{aligned}
 \sum_{i+k \leq z} Q(k, i) &\sim \sum_{i=0}^z \binom{s+i}{s} 2^{-(i+s+1)} \left[1 - \sum_{k=0}^s \binom{z-i+s+1}{s} 2^{-(z-i+s+1)} \right] \\
 &= \sum_{k=0}^z \binom{z+s+1}{k} 2^{-(z+s+1)} - \sum_{i=0}^z \binom{s+i}{s} \sum_{k=0}^s \binom{z-i+s+1}{k} 2^{-(z+2s+2)} \\
 &= - \sum_{k=z+1}^{z+s+1} \binom{z+2s+2}{k} 2^{-(z+2s+2)} + \sum_{i=0}^{z+s+1} \binom{z+2s+2}{i} 2^{-(z+2s+2)} \\
 &= \sum_{i=0}^z \binom{z+2s+2}{i} 2^{-(z+2s+2)} = \sum_{i=0}^z \binom{2(s+1)+i-1}{i} 2^{-2(s+1)} 2^{-i}.
 \end{aligned}$$